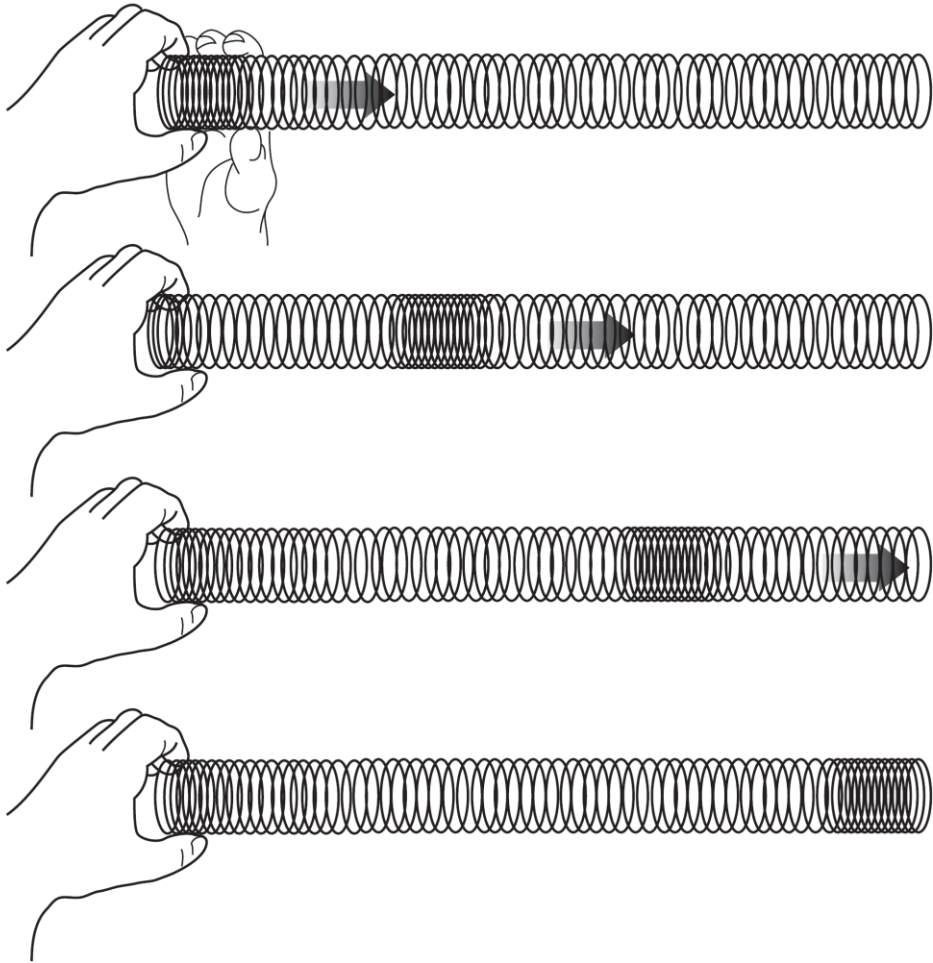
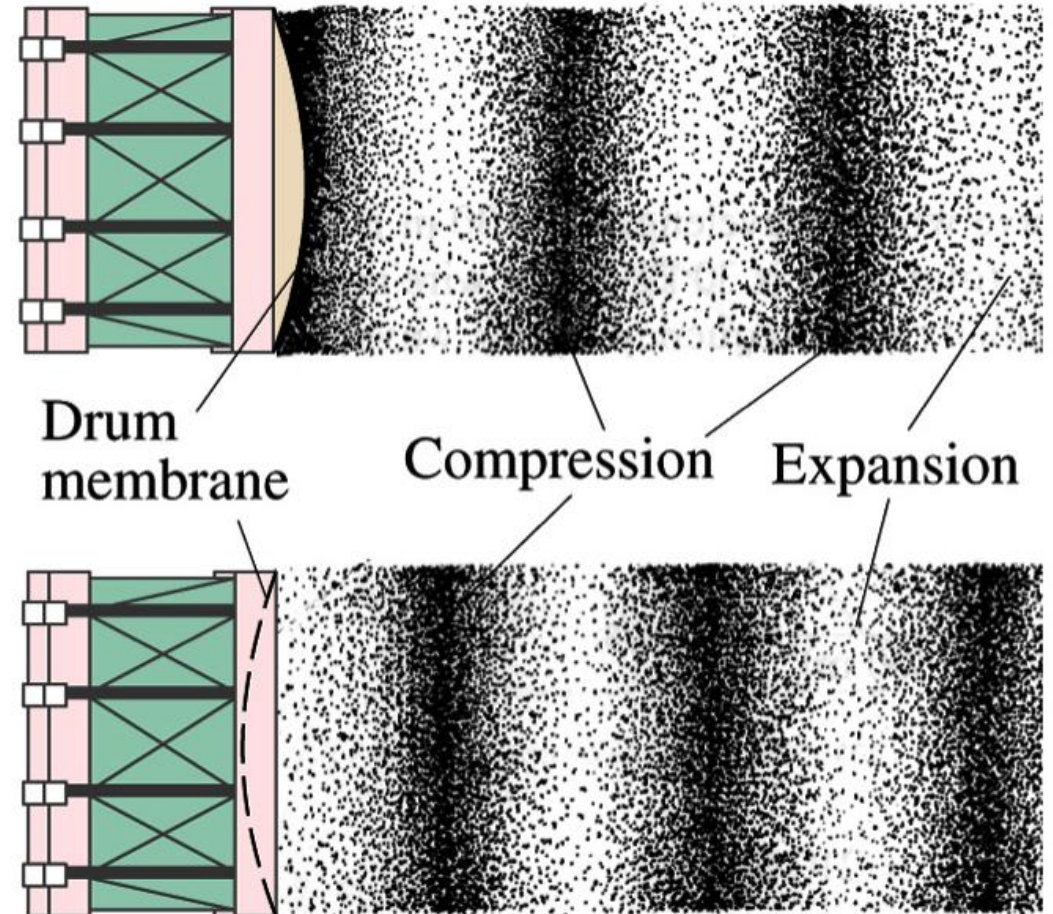


Week 13:
Waves

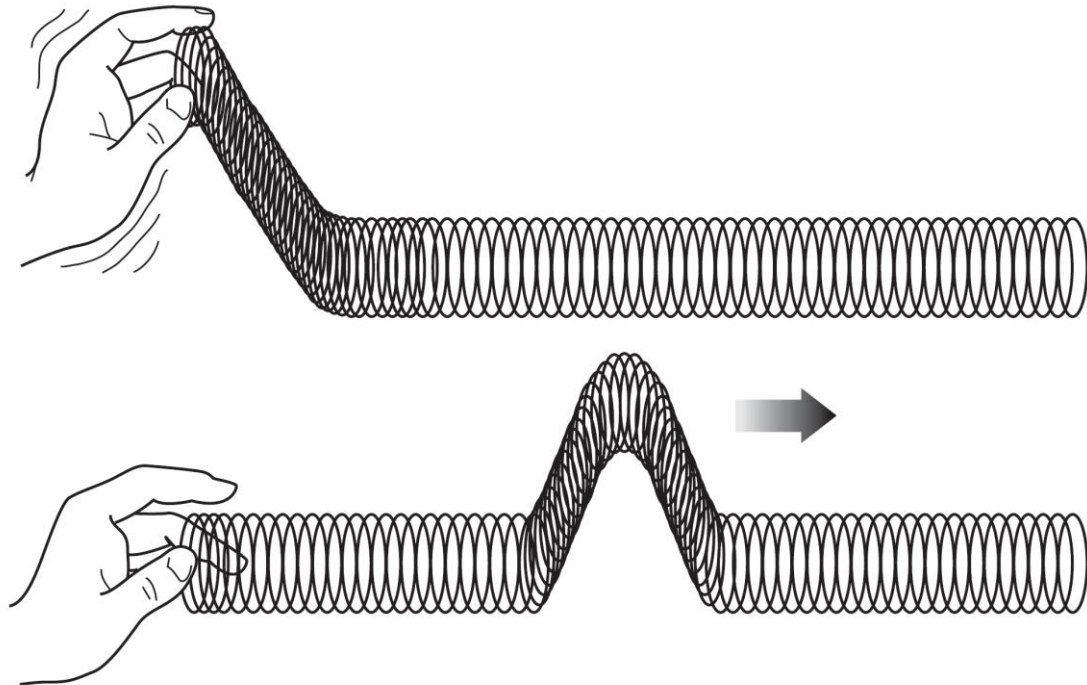
Longitudinal wave in a spring



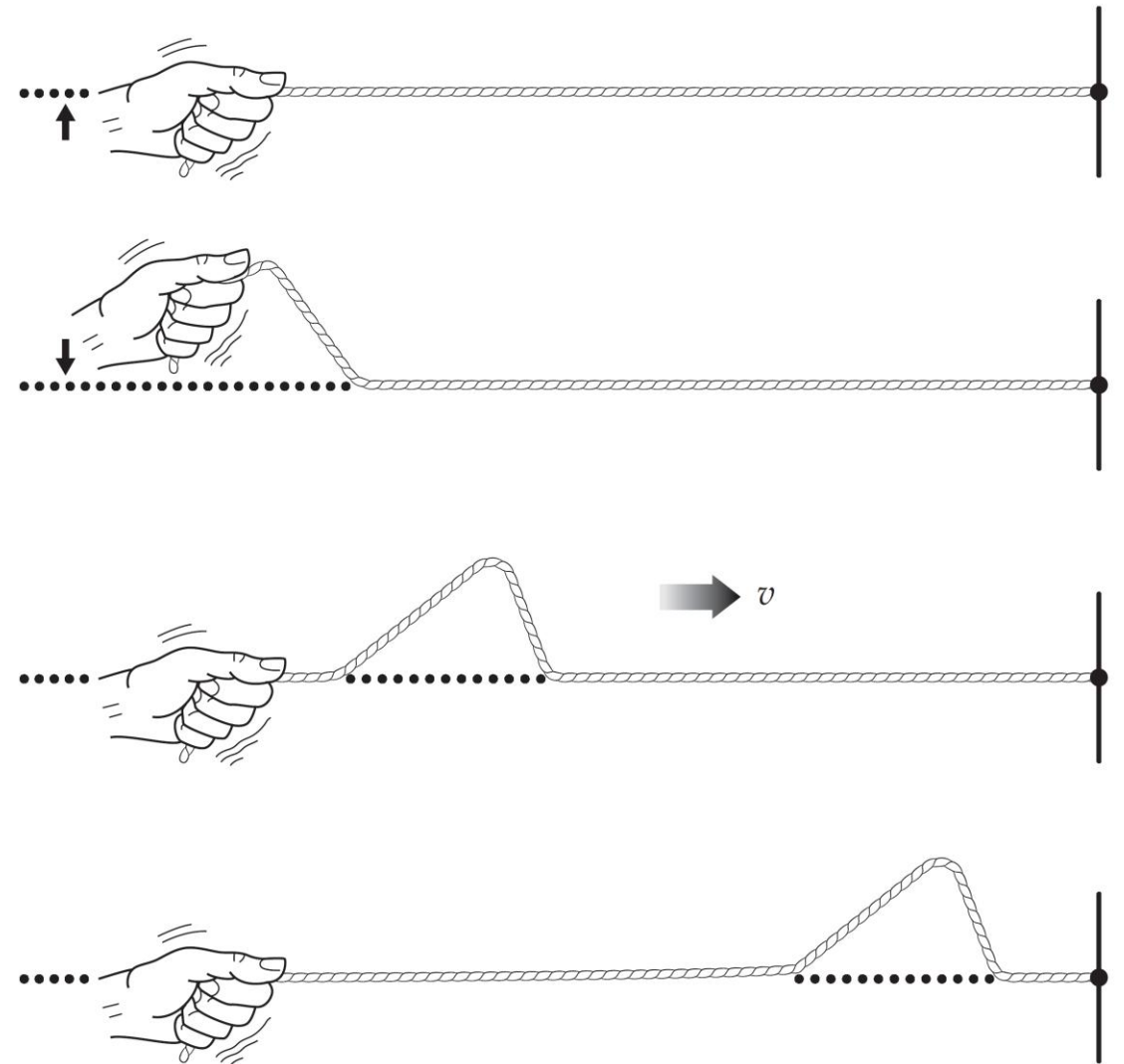
Longitudinal wave in a gas



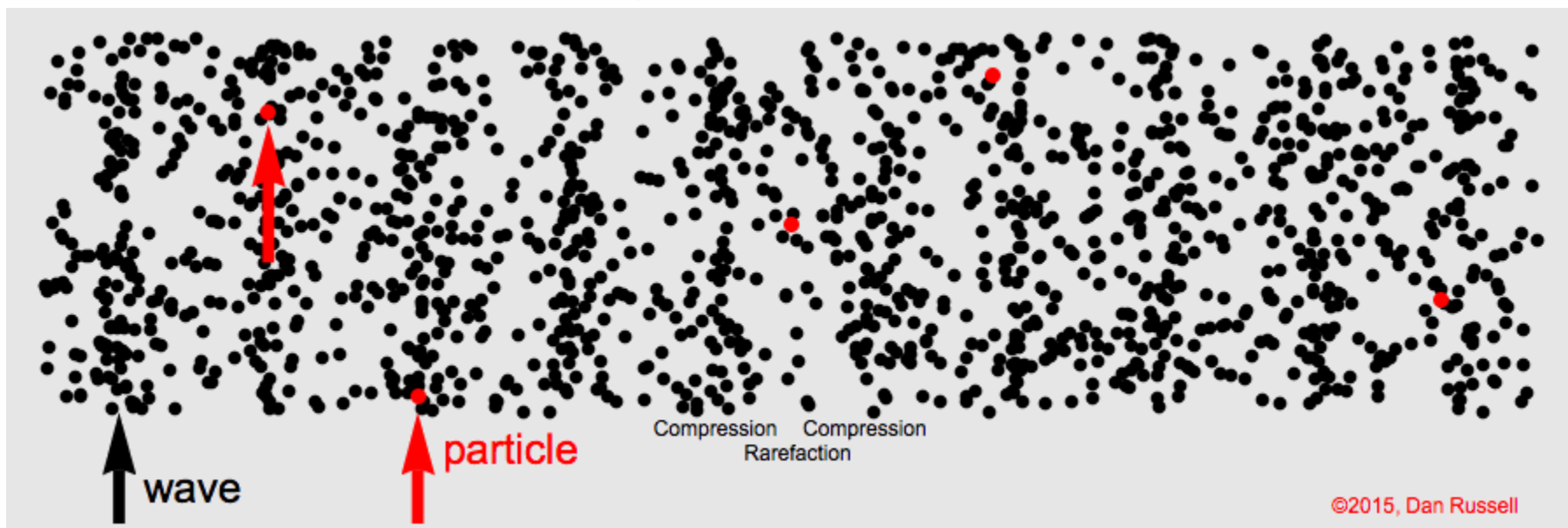
transverse wave in a spring



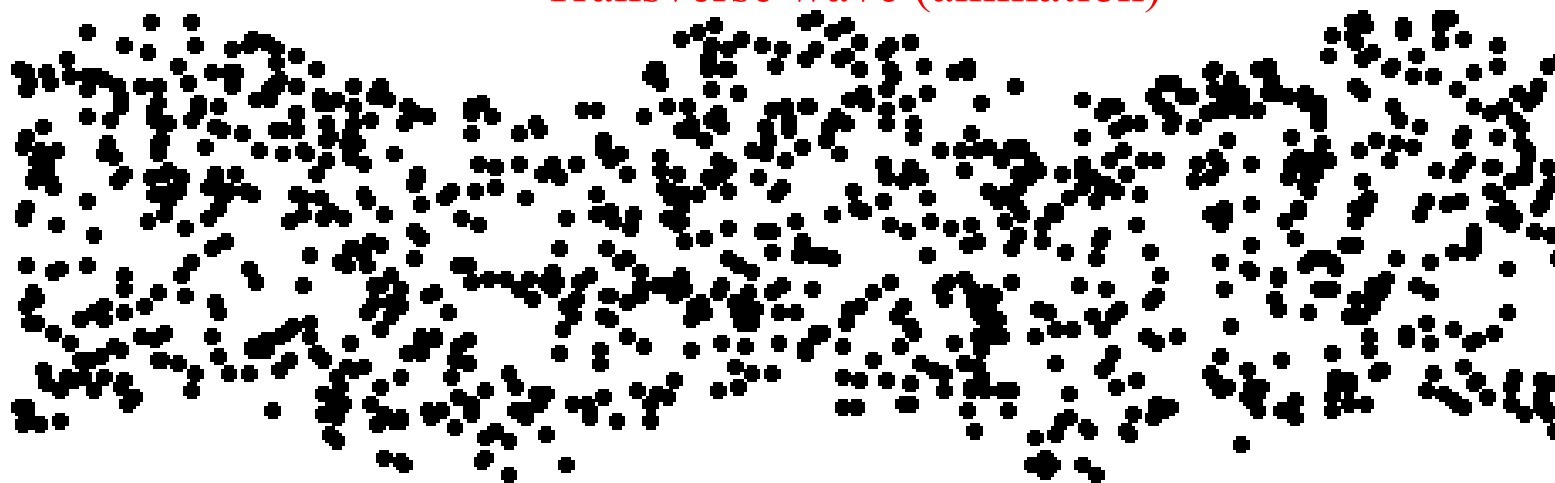
transverse wave in a rope



Longitudinal wave (animation)



Transverse wave (animation)



One-dimensional differential wave equation

Definition: a disturbance $\psi(x, t)$ propagates as a wave without deformation and with a determined v speed if it satisfies:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

Equation of d'Alembert

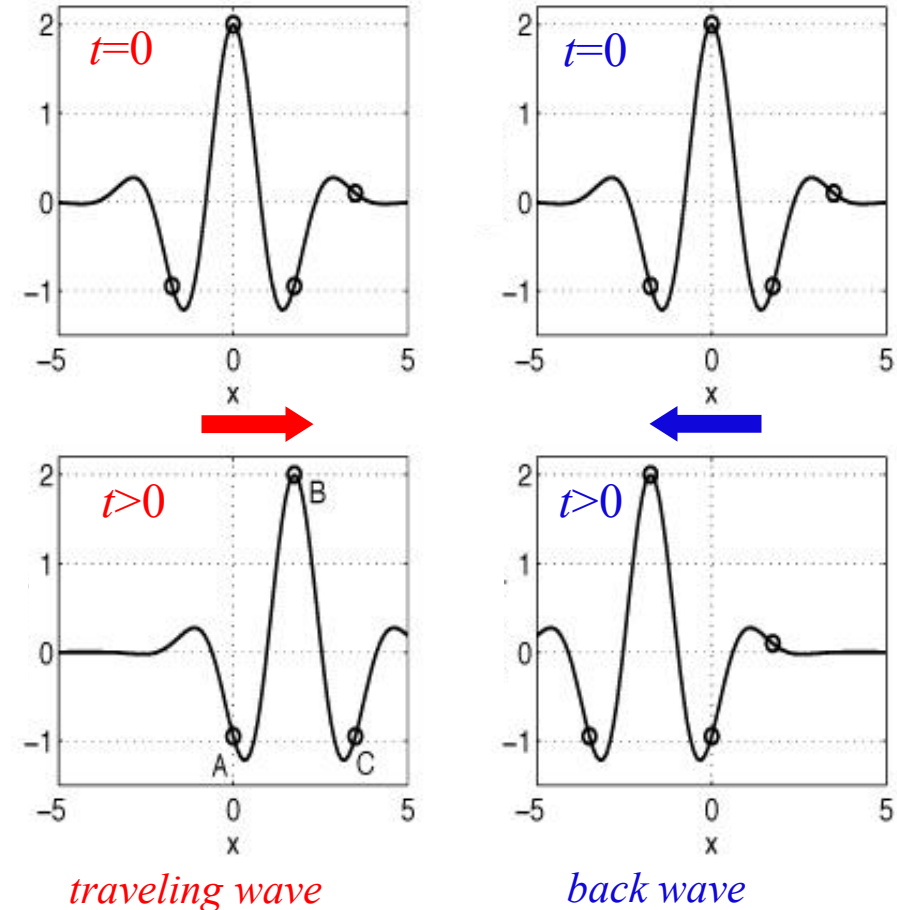
one-dimensional or differential equation of the one-dimensional wave motion

The general solution of the d'Alembert equation is:

$$\psi(x, t) = f(x - vt) + f(x + vt)$$

The d'Alembert wave equation is an example of a linear differential equation, which means that if $\psi_1(x, t)$ and $\psi_2(x, t)$ are solutions to the wave equation, then $\psi_1(x, t) \pm \psi_2(x, t)$ is also a solution.

The implication is that waves solution of the d'Alembert equation (and so also electromagnetic waves, as we will see) obey the superposition principle.



Differential equation of
wave motion
one-dimensional:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

General solution:

$$\psi(x, t) = f(x - vt) + f(x + vt)$$

CURIOSITY

Démonstration:

Let's put: $u = x \pm vt$

\Rightarrow

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial \psi}{\partial u}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial t} = \pm v \frac{\partial \psi}{\partial u}$$

Taking the second derivatives we obtain«

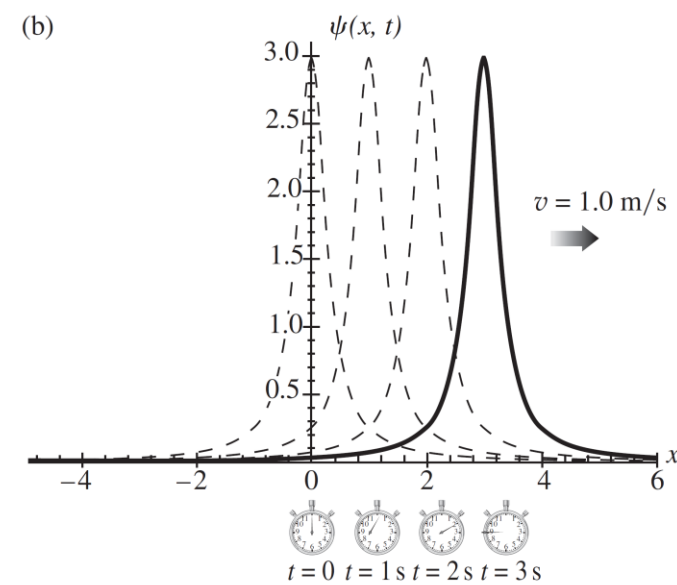
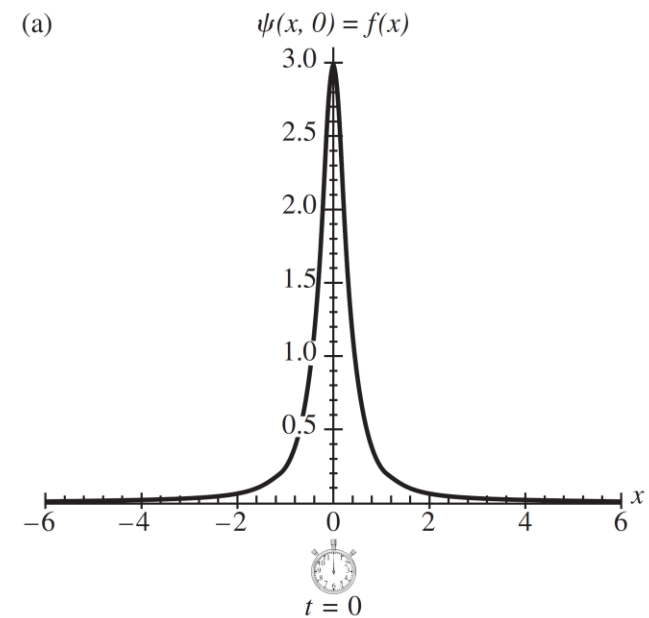
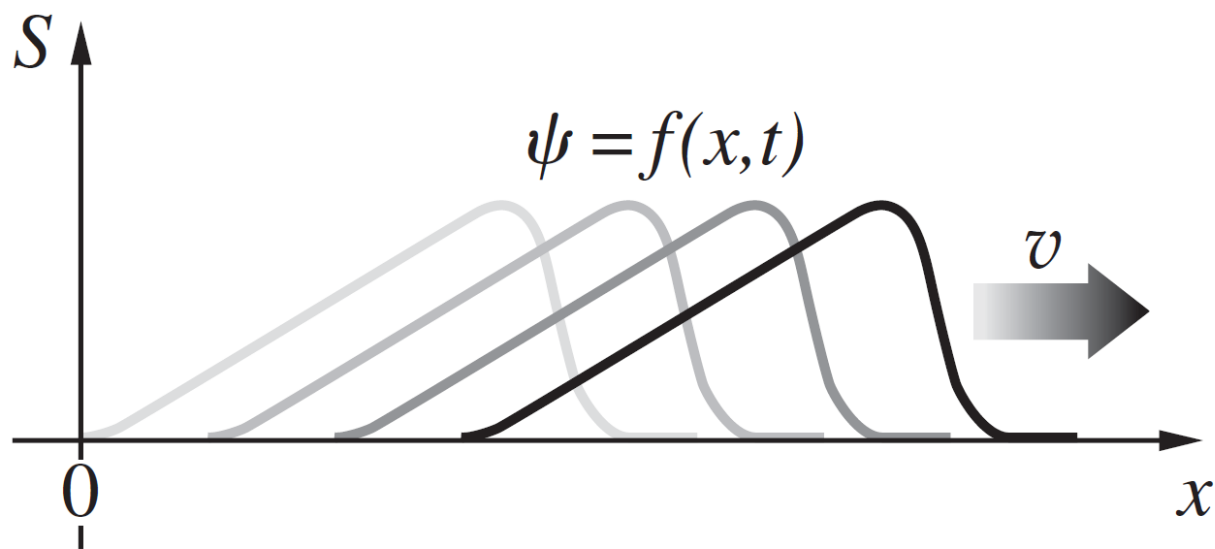
$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial u} \left(\frac{\partial \psi}{\partial x} \right) \frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial u^2}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial u} \frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial u} \left(\frac{\partial \psi}{\partial t} \right) \frac{\partial u}{\partial t} = v^2 \frac{\partial^2 \psi}{\partial u^2}$$

\Rightarrow

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

Propagation of the disturbance without deformation



The three-dimensional wave differential equation

1D

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

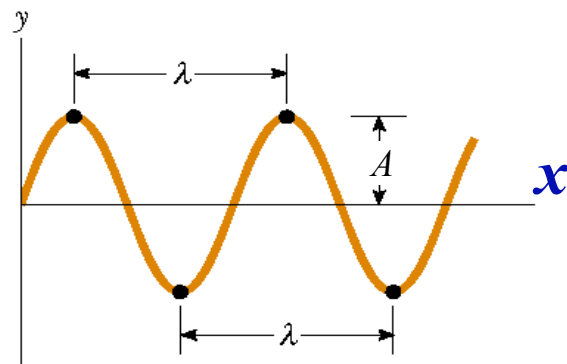
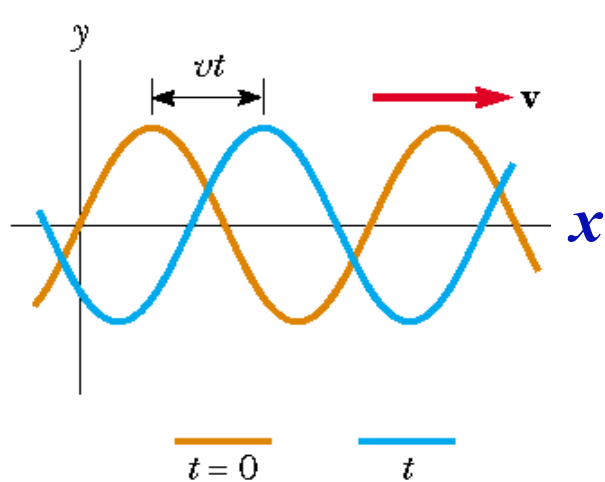
Equation of d'Alembert
one-dimensional or
differential equation of
the wave motion
one-dimensional

3D

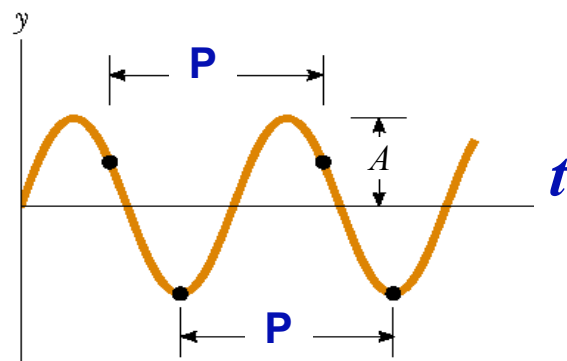
$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = v^2 \nabla^2 \psi$$

Equation of d'Alembert
three-dimensional or
differential equation of
the wave motion
three-dimensional

Sinusoidal plane wave with propagation along x



(a)



Sine wave (equivalent shapes):

$$\begin{aligned}
 \psi(x, t) &= A \sin[k(x \pm vt)] = \\
 &= A \sin\left(kx \pm \frac{2\pi}{\lambda} vt\right) = \\
 &= A \sin\left(\frac{2\pi}{\lambda} (x \pm vt)\right) = \\
 &= A \sin\left(2\pi\left(\frac{x}{\lambda} \pm \frac{t}{P}\right)\right) = \\
 &= A \sin(kx \pm \omega t) \quad (\text{more common form})
 \end{aligned}$$

λ Wavelength[m]

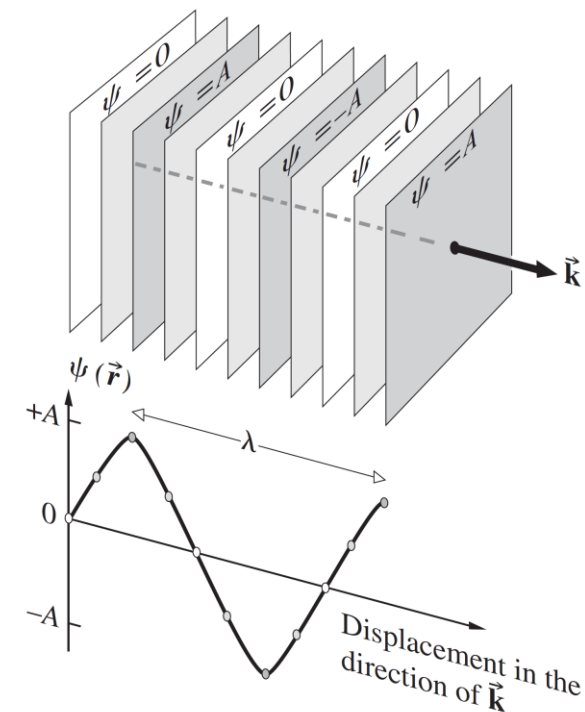
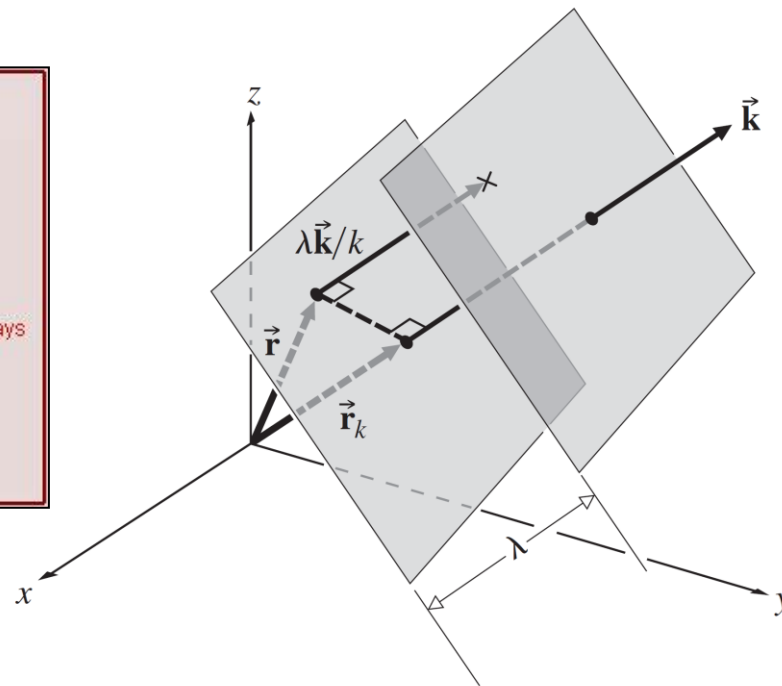
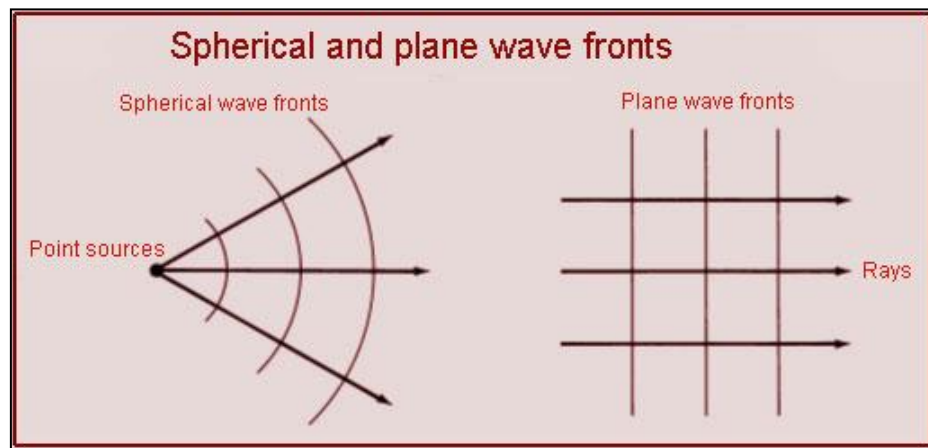
$k = \frac{2\pi}{\lambda}$ Wave number [m^{-1}]

$v = \frac{\Delta x}{\Delta t} = \frac{\lambda}{P} = \lambda f = \frac{\omega}{k}$ phase velocity[m/s]

$f = \frac{v}{\lambda} = \frac{\omega}{2\pi}$ Frequency [Hz]

$P = \frac{1}{f}$ Period [s]

Plane waves with arbitrary propagation direction



Plane sine wave propagating along the axis $\hat{\mathbf{k}}$: $\psi(\mathbf{r}, t) = A \sin(\mathbf{k} \cdot \mathbf{r} \pm \omega t)$

wave-vector \mathbf{k} : $\mathbf{k} = k \hat{\mathbf{k}} \quad k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda} = \frac{\omega}{v}$

\Rightarrow
 $\psi(\mathbf{r}, t) = A \sin(\mathbf{k} \cdot \mathbf{r} \pm \omega t) = A \sin(k_x x + k_y y + k_z z \pm \omega t)$
 (complex form: $\psi(\mathbf{r}, t) = A e^{i(\mathbf{k} \cdot \mathbf{r} \pm \omega t)}$)

Plane waves are a special case of waves where a physical quantity, such as phase, is constant over a plane that is perpendicular to the direction of wave travel.

Electromagnetic Waves

Relation between Electric and Magnetic fields

- **A Change of Magnetic Flux Produces an Electric Field**

The induction of a current in the loop implies the presence of an induced electric field \vec{E} , which must be tangent to the loop because that is the direction in which the charges in the wire move in response to the electric force.

Kirchhoff's rule:

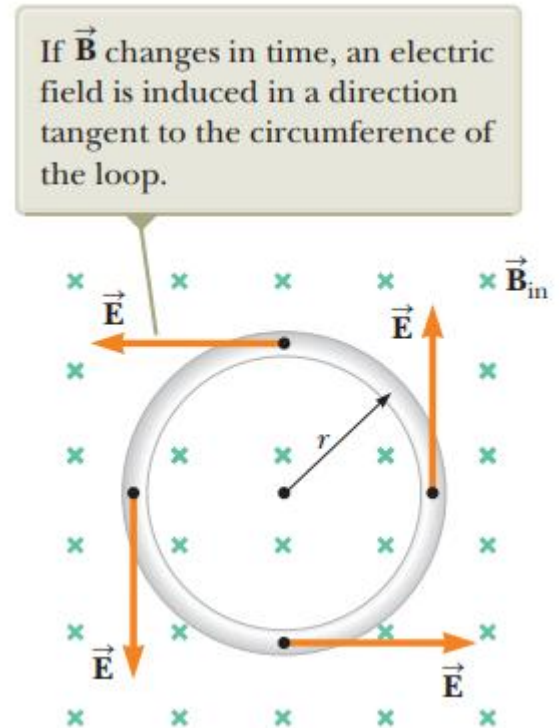
The emf for any closed path can be expressed as the line integral of $\vec{E} \cdot d\vec{\ell}$ over that path

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

Faraday's law

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$



This is a generalization of Faraday's law.

The electric field will exist regardless of whether there are any conductors around.

The induced electric field \vec{E} is a **nonconservative** field that is generated by a changing magnetic field. The field \vec{E} that satisfies the generalization of Faraday's law cannot possibly be an electrostatic field because were the field electrostatic and hence conservative, the line integral over a closed loop would be zero.

Displacement Current and the General Form of Ampère's Law

Can a changing in Electric Fields produce a Magnetic Fields ?

That a magnetic field is produced by an electric current was discovered by Oersted, and the mathematic relation is given by **Ampère's law**

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I_{\text{encl}}$$

In this equation, the line integral is over any closed path through which conduction current passes, where conduction current is defined by the expression $I=dq/dt$.

Is it possible that magnetic fields could be produced in another way as well?

If a changing magnetic field produces an electric field, then perhaps the reverse might be true as well: that a changing electric field will produce a magnetic field.

If this were true, it would signify a beautiful symmetry in nature.

We now show that Ampère's law in this form is valid only if any electric fields present are constant in time.

James Clerk Maxwell recognized this limitation and modified **Ampère's law to include time-varying electric fields**

Displacement Current and the General Form of Ampère's Law

To back up this idea that a changing electric field might produce a magnetic field, we use the following argument:

- According to Ampère's law, we divide any chosen closed path into short segments and take the dot product of each $d\mathbf{l}$ with the magnetic field at that segment \mathbf{B} , and sum (integrate) all these products over the chosen closed path.
- That sum will equal the total current I that passes through a surface bounded by the path of the line integral (I_{encl}).
- When we applied Ampère's law to the field around a straight wire, we imagined the current as passing through the circular area enclosed by our circular loop, and that area is the flat Surface 1 shown in Figure.
- However, we could just as well use the sack-shaped Surface 2 in Figure as the surface for Ampère's law, since the same current I passes through it.

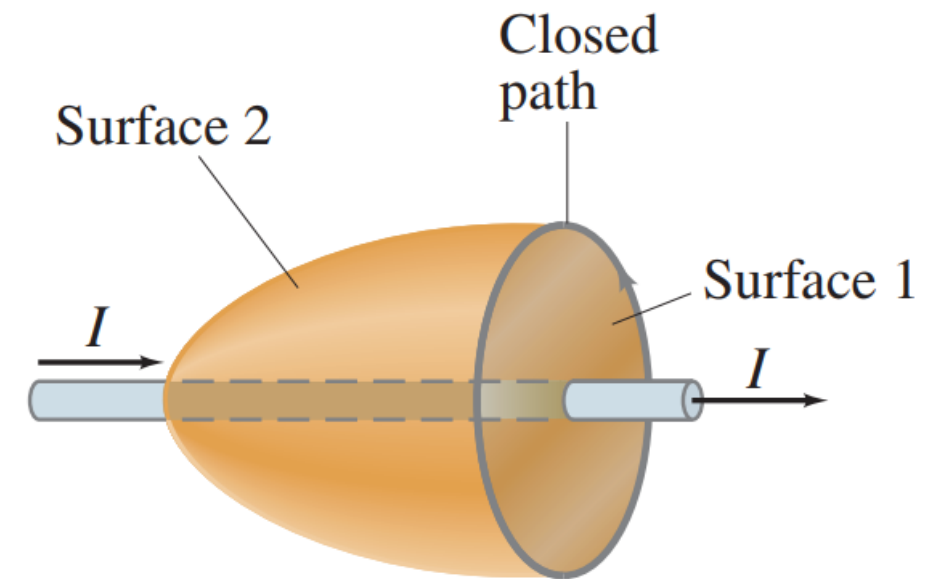


FIGURE 2 Ampère's law applied to two different surfaces bounded by the same closed path.

Displacement Current and the General Form of Ampère's Law

Now consider the closed circular path for the situation of Figure, where a capacitor is being discharged.

Ampère's law works for Surface 1 (current I passes through Surface 1), but it **does not work for Surface 2**, since **no current passes through Surface 2**.

There is a magnetic field around the wire, so the left side of Ampère's law is not zero; yet no current flows through surface 2, so the right side of Ampère's law is zero.

We seem to have a contradiction of Ampère's law !

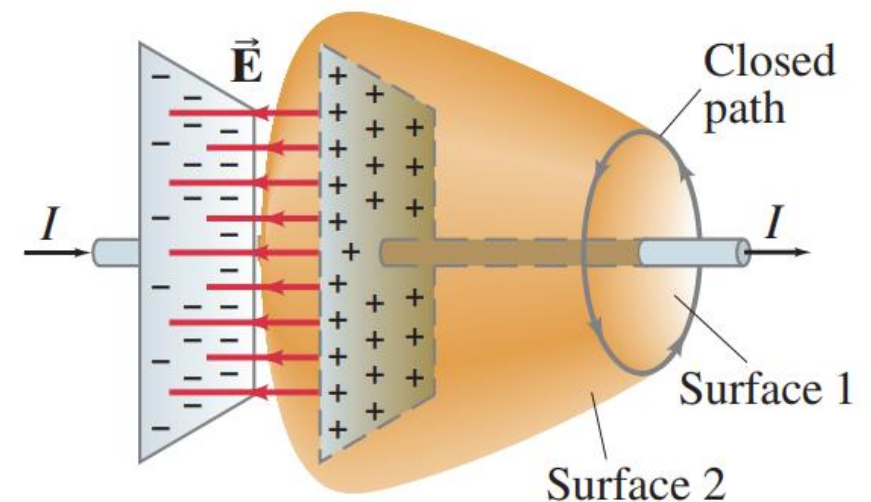
There is a magnetic field present in Figure, however, only if charge is flowing to or away from the capacitor plates.

The changing charge on the plates means that the electric field between the plates is changing in time.

Maxwell resolved the problem of no current through Surface 2 in Figure by proposing that there needs to be **an extra term** on the right in Ampère's law **involving the changing electric field**.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I_{\text{encl}}$$

FIGURE 3 A capacitor discharging. A conduction current passes through surface 1, but no conduction current passes through surface 2. An extra term is needed in Ampère's law.



Displacement Current and the General Form of Ampère's Law

Let us see what this term should be by determining it for the changing electric field between the capacitor plates in Figure.

The charge Q on a capacitor of capacitance C is where V is the potential difference between the plates. Also recall that $V=Ed$, where d is the (small) separation of the plates and E is the (uniform) electric field strength between them, if we ignore any fringing of the field.

Also, for a parallel-plate capacitor, $C = \epsilon_0 A/d$, where A is the area of each plate.

We combine these to obtain

$$Q = CV = \left(\epsilon_0 \frac{A}{d} \right) (Ed) = \epsilon_0 AE$$

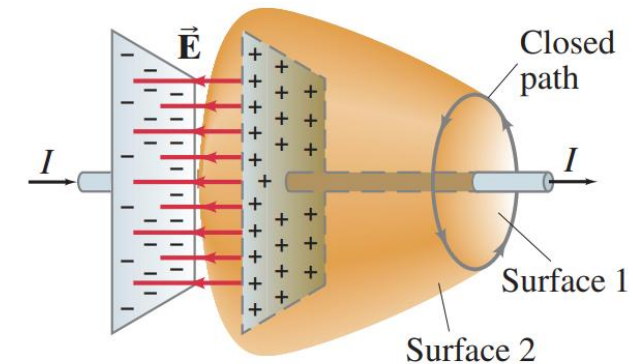
If the charge on each plate changes at a rate dQ/dt , the electric field changes at a proportional rate. That is, by differentiating this expression for Q , we have:

Now is also the current I flowing into or out of the capacitor:

$$\frac{dQ}{dt} = \epsilon_0 A \frac{dE}{dt}$$

$$I = \frac{dQ}{dt} = \epsilon_0 A \frac{dE}{dt} = \epsilon_0 \frac{d\Phi_E}{dt}$$

Where $\Phi_E = EA$ is the electric flux through the closed path (**Surface 2**)

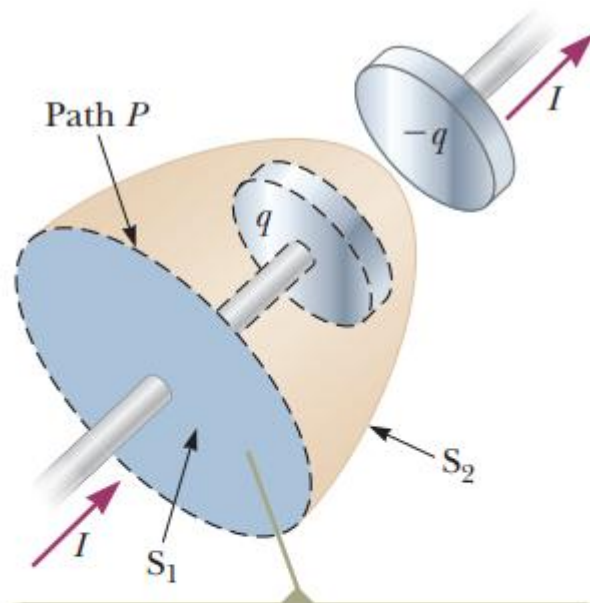


Displacement Current and the General Form of Ampère's Law

In order to make Ampère's law working for surface S_2 in Figure, as well as for surface S_1 (where current I flows), we therefore write

$I_d =$ displacement current

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I_{\text{encl}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad \text{Ampere's law (general form)}$$



The conduction current I in the wire passes only through S_1 , which leads to a contradiction in Ampère's law that is resolved only if one postulates a displacement current through S_2 .

This equation represents the **general form of Ampère's law**, and embodies Maxwell's idea that **a magnetic field can be caused not only by an ordinary electric current, but also by a changing electric field or changing electric flux.**

Although we arrived at it for a special case, this relation has proved valid in general. The last term on the right of this equation is usually very small, and not easy to measure experimentally

As the capacitor is being charged (or discharged), the changing electric field between the plates may be considered equivalent to a current that acts as a continuation of the conduction current in the wire.

Displacement Current and the General Form of Ampère's Law

The electric field lines between the plates create an electric flux through surface S .

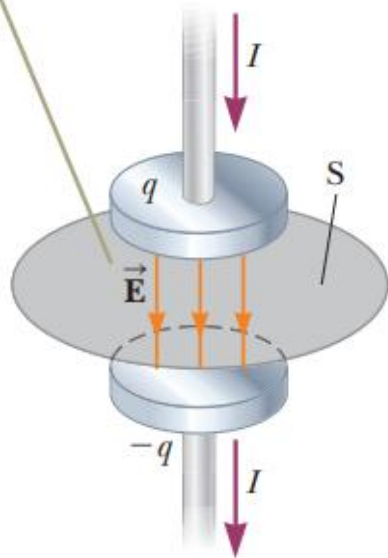


Figure 34.2 When a conduction current exists in the wires, a changing electric field \vec{E} exists between the plates of the capacitor.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0(I + I_d) = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

We can understand the meaning of this expression by referring to Figure 34.2. The electric flux through surface S is $\Phi_E = \int \vec{E} \cdot d\vec{A} = EA$, where A is the area of the capacitor plates and E is the magnitude of the uniform electric field between the plates. If q is the charge on the plates at any instant, then $E = q/(\epsilon_0 A)$

Therefore, the electric flux through S is

$$\Phi_E = EA = \frac{q}{\epsilon_0}$$

Hence, the displacement current through S is

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} = \frac{dq}{dt}$$

That is, the displacement current I_d through S is precisely equal to the conduction current I in the wires connected to the capacitor!

EXAMPLE 1 Charging capacitor. A 30-pF air-gap capacitor has circular plates of area $A = 100 \text{ cm}^2$. It is charged by a 70-V battery through a $2.0\text{-}\Omega$ resistor. At the instant the battery is connected, the electric field between the plates is changing most rapidly. At this instant, calculate (a) the current into the plates, and (b) the rate of change of electric field between the plates. (c) Determine the magnetic field induced between the plates. Assume \vec{E} is uniform between the plates at any instant and is zero at all points beyond the edges of the plates.

APPROACH In RC circuits, the charge on a capacitor being charged, as a function of time, is

$$Q = CV_0(1 - e^{-t/RC}),$$

where V_0 is the voltage of the battery. To find the current at $t = 0$, we differentiate this and substitute the values $V_0 = 70 \text{ V}$, $C = 30 \text{ pF}$, $R = 2.0 \text{ }\Omega$.

SOLUTION (a) We take the derivative of Q and evaluate it at $t = 0$:

$$\left. \frac{dQ}{dt} \right|_{t=0} = \left. \frac{CV_0}{RC} e^{-t/RC} \right|_{t=0} = \frac{V_0}{R} = \frac{70 \text{ V}}{2.0 \text{ }\Omega} = 35 \text{ A}.$$

This is the rate at which charge accumulates on the capacitor and equals the current flowing in the circuit at $t = 0$.

(b) The electric field between two closely spaced conductors is given by

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0}.$$

Hence

$$\frac{dE}{dt} = \frac{dQ/dt}{\epsilon_0 A} = \frac{35 \text{ A}}{(8.85 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)(1.0 \times 10^{-2} \text{ m}^2)} = 4.0 \times 10^{14} \text{ V/m}\cdot\text{s}.$$

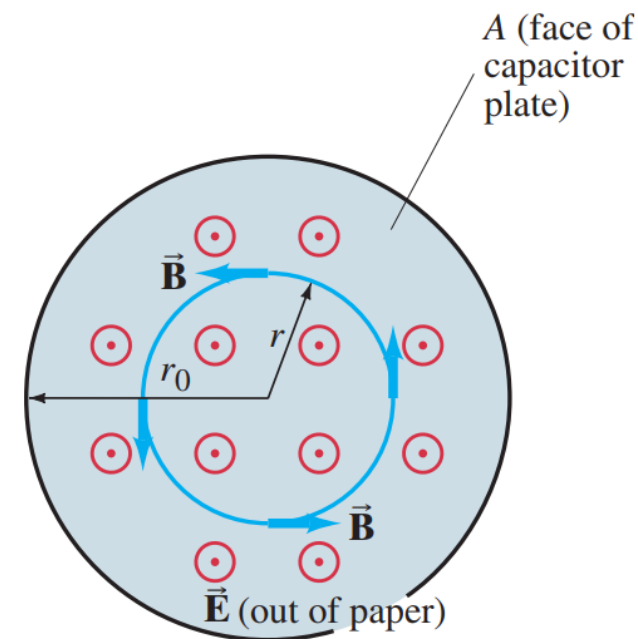


FIGURE 4 Frontal view of a circular plate of a parallel-plate capacitor. \vec{E} between plates points out toward viewer; lines of \vec{B} are circles. (Example 1.)

(c) Although we will not prove it, we might expect the lines of $\vec{\mathbf{B}}$, because of *symmetry*, to be circles, and to be perpendicular to $\vec{\mathbf{E}}$, as shown in Fig. 4; this is the same symmetry we saw for the inverse situation of a changing magnetic field producing an electric field. To determine the magnitude of B between the plates we apply Ampère's law, Eq. 1, with the current $I_{\text{encl}} = 0$:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

We choose our path to be a circle of radius r , centered at the center of the plate, and thus following a magnetic field line such as the one shown in Fig. 4. For $r \leq r_0$ (the radius of plate) the flux through a circle of radius r is $E(\pi r^2)$ since E is assumed uniform between the plates at any moment. So from Ampère's law we have

$$\begin{aligned} B(2\pi r) &= \mu_0 \epsilon_0 \frac{d}{dt} (\pi r^2 E) \\ &= \mu_0 \epsilon_0 \pi r^2 \frac{dE}{dt}. \end{aligned}$$

Hence

$$B = \frac{\mu_0 \epsilon_0}{2} r \frac{dE}{dt}. \quad [r \leq r_0]$$

Thus the B field outside the capacitor is the same as that outside the wire. In other words, the magnetic field produced by the changing electric field between the plates is the same as that produced by the current in the wire

$$r > r_0: \quad E = \sigma/\epsilon_0 = Q/(\epsilon_0 A), \quad dE/dt = I/(\epsilon_0 A)$$

$$B = \frac{\mu_0 \epsilon_0 r_0^2}{2r} \frac{dE}{dt} = \frac{\mu_0 \epsilon_0 r_0^2}{2r} \frac{I}{\epsilon_0 \pi r_0^2} = \frac{\mu_0 I}{2\pi r}$$

We assume $\vec{\mathbf{E}} = 0$ for $r > r_0$, so for points beyond the edge of the plates all the flux is contained within the plates (area = πr_0^2) and $\Phi_E = E\pi r_0^2$. Thus Ampère's law gives

$$\begin{aligned} B(2\pi r) &= \mu_0 \epsilon_0 \frac{d}{dt} (\pi r_0^2 E) \\ &= \mu_0 \epsilon_0 \pi r_0^2 \frac{dE}{dt} \end{aligned}$$

or

$$B = \frac{\mu_0 \epsilon_0 r_0^2}{2r} \frac{dE}{dt}. \quad [r \geq r_0]$$

B has its maximum value at $r = r_0$ which, from either relation above (using $r_0 = \sqrt{A/\pi} = 5.6$ cm), is

$$\begin{aligned} B_{\text{max}} &= \frac{\mu_0 \epsilon_0 r_0}{2} \frac{dE}{dt} \\ &= \frac{1}{2} (4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}) (8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2) (5.6 \times 10^{-2} \text{ m}) (4.0 \times 10^{14} \text{ V/m} \cdot \text{s}) \\ &= 1.2 \times 10^{-4} \text{ T}. \end{aligned}$$

This is a very small field and lasts only briefly (the time constant $RC = 6.0 \times 10^{-11}$ s) and so would be very difficult to measure.

The magnetic field lines generated by a current and of a bar magnet do not begin or end at any point. **For any closed surface, the number of lines entering the surface equals the number leaving the surface; therefore, the net magnetic flux is zero.** In contrast, for a closed surface surrounding one charge of an electric, the net electric flux is not zero.

Gauss's law in magnetism states that

the net magnetic flux through any closed surface is always zero:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

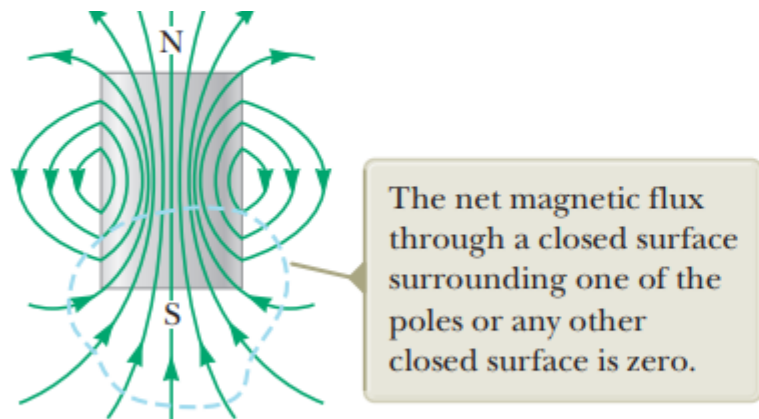


Figure 30.22 The magnetic field lines of a bar magnet form closed loops. (The dashed line represents the intersection of a closed surface with the page.)

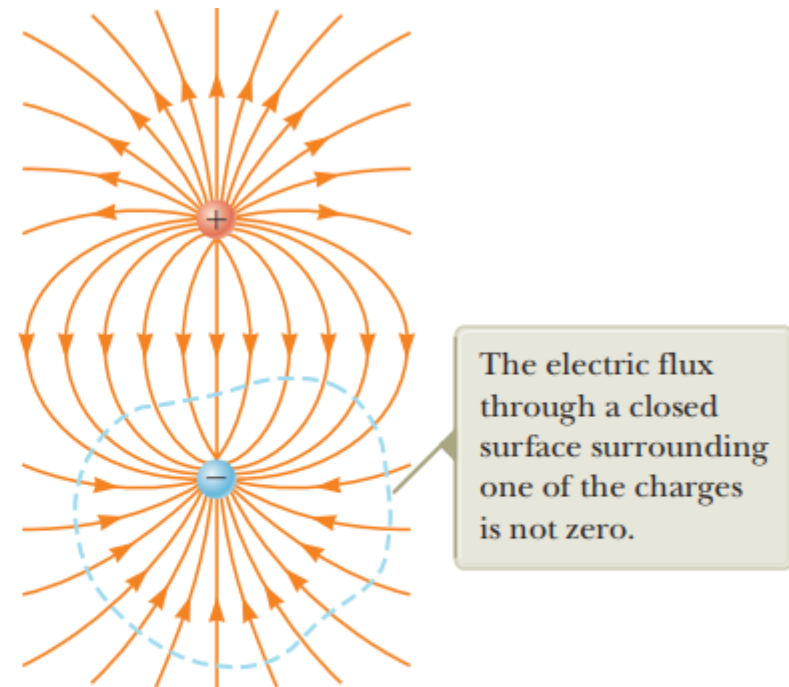


Figure 30.23 The electric field lines surrounding an electric dipole begin on the positive charge and terminate on the negative charge.

Maxwell's Equations (integral form)

We now present **four equations** that are regarded as the basis of all electrical and magnetic phenomena. These equations, developed by Maxwell, are as fundamental to electromagnetic phenomena as Newton's laws are to mechanical phenomena.

The theory that Maxwell developed turned out to also be in agreement with the special theory of relativity, as Einstein showed in 1905.

Maxwell's equations represent the laws of electricity and magnetism that we have already discussed, but they have additional important consequences. For simplicity, we present Maxwell's equations as applied to free space, that is, in the absence of any dielectric or magnetic material. The four equations are:

$$(1) \quad \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q}{\epsilon_0}$$

◀ Gauss's law

$$(2) \quad \oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

◀ Gauss's law in magnetism

$$(3) \quad \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

◀ Faraday's law

$$(4) \quad \oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

◀ Ampère–Maxwell law

Notice the symmetry of Maxwell's equations. Equations (1) and (2) are symmetric, apart from the absence of the term for magnetic monopoles in Equation (2). Furthermore, Equations (3) and (4) are symmetric in that the line integrals of \mathbf{E} and \mathbf{B} around a closed path are related to the rate of change of magnetic flux and electric flux, respectively.